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A CONDITION FOR ZERO ENTROPY

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ABSTRACT

Criteria for a homeomorphism of a compact metric space to have zero topological entropy are obtained. These are applied to show that minimal distal and POD flows have zero entropy. These notions are also relativized, and it is shown that entropy is preserved by the extensions they define.

Parry [3] introduced the notions of a separating point and a separating sieve to provide an easy proof that the entropy of a distal action of the integers on a compact metric space is zero. His methods were designed to apply to distal or similar actions, and the existence of a separating point or sieve implies the existence of a measurable eigenfunction. We will introduce a variant notion which is more broadly applicable. Some of the entropies calculated are known but the suggested notions provide simple proofs, all of a similar form, based on properties which appear to be common and interesting. We will suppose throughout that T is a homeomorphism of the compact metric space X onto itself.

DEFINITION. We say $x \in X$ is a p-point if whenever y and z are distinct points of X there is a sequence of integers n_i satisfying either $T^{n_i}y \to x$ and $T^{n_i}z \not\to x$ or vice-versa.

Although the existence of a p-point does not imply minimality, there could not be three minimal subsets and if there are two minimal subsets then at least one of them must be a singleton set. Our next definition is somewhat less restrictive.

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DEFINITION. Suppose $\mathscr{E} = \{E_1, E_2, \dots\}$ is a sequence of Borel subsets of X. We say $\mathscr{E}$ is a separator if:

(1) $\limsup_{n\to\infty} E_n$ has invariant measure zero and

(2) If $\mathscr{E}'$ is an infinite sub-collection of sets from $\mathscr{E}$ and x and y are distinct points of X then there is an integer k and a set $E \in \mathscr{E}'$ which separates $T^k x$ from $T^k y$.

We point out that the integer k may be positive, negative or zero, the conditions on $\mathscr{E}$ are independent of the order of enumeration of the sets, and that an infinite sub-collection of a separator is a separator. Further, the definition is applicable for Borel isomorphisms on a standard Borel space.

DEFINITION. A collection of sets $\mathscr{N}$ is called a stable base for the neighborhood system of a point $x \in X$ if every infinite sub-collection from $\mathscr{N}$ is a base for the neighborhood system of x.

LEMMA. *If x is a p-point whose orbit is infinite and $\mathscr{E} = \{E_1, E_2, \dots\}$ is a countable stable base for the neighborhood system of x then $\mathscr{E}$ is a separator.*

The proof is an immediate consequence of the definitions.

THEOREM. *If (X, T) admits a separator $\mathscr{E}$ then the topological entropy of T is zero.*

PROOF. It suffices to show that the measure theoretic entropy is zero for every ergodic invariant probability measure. Thus let μ be an ergodic invariant measure. We use the separator to construct a countable generating partition of arbitrarily small entropy. First set $M = \limsup_{n\to\infty} E_n$, so $\mu(M) = 0$. If $x \notin M$ then x is in only finitely many of the members of $\mathscr{E}$. For each finite subset J of $\mathbf{N}$ we set $F_J = \{x \mid x \in E_j \text{ iff } j \in J\}$. We include the void subset $J = \varnothing$ and observe $F_\varnothing = \{x \mid x \notin E_j\ \forall j\}$. It is clear the collection $\mathscr{F} = \{\mathscr{F}_J\}$ partitions $X \setminus M$, and $\mathscr{F}$ generates because $\mathscr{E}$ is a separator. The μ entropy of the transformation T is thus bounded by the μ entropy of the partition $\mathscr{F}$.

We let $\eta(t) = -t\log(t)$. Let $\varepsilon > 0$. We can suppose the separator has been renumbered so $\mu(E_n)$ is monotonically decreasing (to zero), and we can suppose $\mathscr{E}$ has been replaced by an infinite sub-collection such that η is monotone on the interval $[0, \mu(E_1)]$, and that $\eta(t) < \varepsilon$ if $t \geqq \mu(X \setminus \bigcup_{k>0} E_k)$. If J is a non-void finite subset of $\mathbf{N}$ let $|J|$ denote its largest element. Then $\mu(F_J) \leqq \mu(E_{|J|})$ and the entropy of $\mathscr{F}$ is bounded by $\eta(\mu(J_\varnothing)) + \Sigma_{n>0} 2^{n-1}\eta(\mu(E_n))$. We replace the separator by a new one, with $\mu(E_n) \to 0$ so rapidly that the sum is less than ε. □

OBSERVATIONS. (1) If T is minimal and distal then $\lim T^{n_i}x = \lim T^{n_i}y$ implies $x = y$. Consequently every point is a p-point. Thus a distal minimal action has zero entropy.

(2) Suppose that X is minimal and whenever $x \neq y$ the closure of the orbit of (x, y) in the product space $X \times X$ contains a minimal subset other than the diagonal. Then every point of X is a p-point. (Pr: Let $x \neq y$ and let z be arbitrary. There is an almost periodic point (u, v) in the orbit closure of (x, y) with $u \neq v$. Then z is in the orbit closure of u, and there must be a $w \neq z$ such that (w, z) is in the orbit closure of (u, v) and so in the orbit closure of (x, y).) Recall that the syndetically proximal relation is the subset L of $X \times X$ such that $(x, y) \in L$ iff whenever (x', y') is in the orbit closure of (x, y) then x' and y' are proximal. Using this language, we have shown the entropy is zero if the minimal action T has a trivial syndetically proximal relation.

(3) A totally minimal action T is called POD (proximal orbit dense) [2] if whenever $x \neq y$ the orbit closure of (x, y) contains a graph $G_n = \{(u, v) \mid v = T^n u\}$ for some $n \neq 0$. Thus POD implies zero entropy.

(4) The existence of a p-point y implies the function $f(x) = d(x, y)$ "separates orbits", i.e.: If $x \neq z$ then $f(T^n x) \neq f(T^n z)$ for some n. The existence of such a "Bebutov function" implies the flow (X, T) can be represented by translation in the space of bounded functions on $\mathbf{Z}$, with the topology of uniform convergence on compact (= finite) subsets.

(5) We formulate explicitly a contrapositive result: If a minimal transformation T has positive entropy then for every point x there are distinct points y and z such that $T^{n_i}y \to x$ iff $T^{n_i}z \to x$.

A minimal action also described in [1] has a separator but no p-points. Let R be a closed rectangle in the plane with sides parallel to the coordinate axes. We construct three new rectangles R_0, R_1, R_2 contained in R as follows. Each R_i is to have one-fifth the width of R. R_0 and R_2 have one-half the height of R, and R_1 has the same height as R. The lower left corner of R is the lower left corner of R_0, the upper right corner of R is the upper right corner of R_2, and the midpoint of the base of R is the midpoint of the base of R_1. Begin with, say, the unit square R and inductively construct rectangles $R_{i_1,i_2,\ldots,i_k}$ with each index $i \in \{0, 1, 2\}$. We set $X_k = \bigcup \{R_{i_1,i_2,\ldots,i_k} \mid (i_1, i_2, \ldots, i_k) \in \{0, 1, 2\}^k\}$ and $X = \bigcap X_k$. A point $x \in X$ determines an element $\Pi(x) \in \{0, 1, 2\}^N$ by $\Pi(x) = i_1, i_2, \ldots$ iff $x \in R_{i_1,i_2,\ldots,i_k}$ $\forall k$. If we set $R_{i_1,i_2,\ldots} = \bigcap_k R_{i_1,i_2,\ldots,i_k}$ then $x \in R_{\Pi(x)}$ and $R_{i_1,i_2,\ldots}$ is a rectangle with zero width and height $(\frac{1}{2})^{\#\text{ indices different from }1}$ ($= 0$ if the number of such indices is infinite). We define a mapping T on X by lifting the "odometer mapping" of $\{0, 1, 2\}^N$ (add one on the left, carry to the right) so that the vertical lines are mapped linearly onto each other. It is not difficult to show T is a minimal homeomorphism of X and Π is a continuous mapping intertwining with T and the odometer mapping.

If $\Pi(y) = \Pi(z)$, $y \neq z$ and (u, v) is in the orbit closure of (x, y) then $\Pi(u) = \Pi(v)$. It follows that any potential p-point must lie on a non-trivial vertical line segment in X. Let x be such a point. Take a non-trivial vertical line segment L in X and choose two points $y \in L$ and $z \in L$. We can choose these points so that no points in their orbits are at exactly the same heights as the point x. Any maximal vertical segment M in X must be of height $(\frac{1}{2})^k$ and must have its base at height $j/2^k$ for some j, k. This fact, the linearity of T on segments, and the choice of y, imply that $T^{n_i}(y)$ can only converge to x if the height of the $R_{\Pi(T^{n_i}(y))} \to 0$. But then $T^{n_i}(z) \to x$ also (and vice-versa). Thus there are no p-points.

To construct a separator, let $H_1, H_2, \ldots$ be a sequence of subsets of the unit interval which separates points of $[0, 1]$ and such that every point in the unit interval is in at least one H_k. The rectangle $R_{1,1,1,\ldots}$ is the vertical line L of height one which bisects the rectangle R. We let V_n be the vertical strip of points in R whose horizontal distance from L is less than $1/n$. We can see that every V_n contains rectangles $R_{i_1,i_2,\ldots}$ of length $(\frac{1}{2})^k$ for every $k = 1, 2, 3, \ldots, \infty$, and $\Pi(V_n)$ is a base for the neighborhood system of $\Pi(L)$ in the space $\{0, 1, 2\}^N$. For each n, we construct E_n as a subset of V_n. Let $x \in X \cap V_n$. If $R_{\Pi(x)}$ has height $(\frac{1}{2})^k$ map the unit interval linearly onto $R_{\Pi(x)}$ and place x in E_n if the pre-image of x is in H_k. We observe that when $T^j(x)$ returns to V_n the height of $R_{\pi(T^j(x))}$ may be different but the pre-image under the associated linear mapping will be the same. If $R_{\Pi(x)}$ has zero height, place x in E_n. We now show $\mathscr{E} = \{E_1, E_2, \ldots\}$ is a separator. First suppose $\Pi(x) \neq \Pi(y)$. Because the odometer mapping is distal, we can choose n so that if $T^j x \in V_n$ then $T^j y \notin V_n$. If $R_{\Pi(x)}$ is a point then E_n separates $T^j(x)$ and $T^j(y)$. If $R_{\Pi(x)}$ is a non-trivial interval then the orbit of x passes through all non-trivial vertical intervals. Without loss of generality $x \in V_n$. The pre-image of x is in H_k for some k, so when $T^j(x)$ is in an interval of height $(\frac{1}{2})^k$ which is contained in V_n then $T^j(x) \in E_n$ (and $T^j(y) \notin E_n$). Thus E_n separates $T^j x$ from $T^j y$ for some j. When $\Pi(x) = \Pi(y)$ the separating property of the sets $H_1, H_2, \ldots$ ensures us that any one of the sets E_n will separate $T^j x$ from $T^j y$ for some j.

Alternatively, the definition of a separator could be modified to allow the deletion of a set of invariant measure zero from the space.

The above example has no p-points. It is also possible to construct a flow for which some but not all points are p-points. Simply "split the points" on one orbit of an irrational rotation of a circle. The resulting flow is an almost one to one extension (of the rotation) with a single pair of doubly asymptotic orbits. The p-points are exactly the points on these orbits.

Relative Notions

We now assume S is a homeomorphism of a compact metric space Y onto itself, Π is a continuous mapping of X onto Y, and $S \circ \Pi = \Pi \circ T$.

DEFINITION. We say $x \in X$ is a relative (to Π) p-point if whenever y and z are distinct points of X with $\Pi(y) = \Pi(z)$ there is a sequence of integers n_i satisfying either $T^{n_i}y \to x$ and $T^{n_i}z \not\to x$ or vice-versa.

DEFINITION. Suppose $\mathscr{E} = \{E_1, E_2, \ldots\}$ is a sequence of Borel subsets of X. We say $\mathscr{E}$ is a relative (to π) separator if:

(1) $\lim\sup_{n \to \infty} E_n$ has invariant measure zero and

(2) If $\mathscr{E}'$ is an infinite sub-collection of sets from $\mathscr{E}$ and x and y are distinct points of X with $\Pi(x) = \Pi(y)$ then there is an integer k and a set $E \in \mathscr{E}'$ which separates $T^k x$ from $T^k y$.

LEMMA. *If x is a relative p-point with an infinite orbit and $\mathscr{E} = \{E_1, E_2, \ldots\}$ is a countable stable base for the neighborhood system of x then $\mathscr{E}$ is a relative separator.*

THEOREM. *If (X, T) admits a relative separator $\mathscr{E}$ then the topological entropy of T is equal to the topological entropy of S.*

PROOF. Actually we prove a little more: If μ is an invariant measure on X and $\nu = \mu \circ \Pi^{-1}$ then the μ entropy of T is the same as the ν entropy of S. Since $\mu \to \mu \circ \Pi^{-1}$ is a surjective mapping on the invariant measures, the theorem follows from the identification of the topological entropy as the supremum of the measure theoretic entropies.

Exactly as in the absolute case, we can create a countable partition $\mathscr{F}$ from the separator and, by replacement of the original separator by an infinite sub-collection, we can assume the entropy of the partition $\mathscr{F}$ is as small as we please. Of course $\mathscr{F}$ will not separate points but we can choose an increasingly fine sequence of finite partitions $\mathscr{G}_n$ of Y such that (after deleting a set of measure zero from Y) any pair of distinct points are eventually separated by the partitions $\mathscr{G}_n$. Then (after deleting a set of zero measure from X) if $x \neq x'$ are points in X then either $\Pi(x)$ and $\Pi(x')$ are eventually separated by $\mathscr{G}_n$ or else $\Pi(x) = \Pi(x')$ in which case x and x' are separated by $\mathscr{F}$. Consequently the ν entropy of S is the limit of the conditional entropies of the $\mathscr{G}_n$ on their pasts, and the μ entropy of T is the limit of the conditional entropies of the $(\Pi^{-1}\mathscr{G}_n) \vee \mathscr{F}$ on their pasts. (This can all be said more easily if we use the fact that transformations of finite entropy admit generators of finite entropy. The more involved formulation above keeps the presentation at a more elementary level.) The proof can now be completed by citing routine subadditive properties of entropy. □

We will now construct an example to show that a point may be a relative p-point for a projection and may project to a p-point on the base but not be a p-point in the top space. We first observe that if (x, y) is a p-point for $X \times Y$ then x is a p-point for X. (Given x' and x'' in X, consider (x', y) and (x'', y).) Now choose α and β so that 1, α and β are rationally independent. Let T be the rotation on the circle X by α and let (Y, S) be the almost one to one extension of the rotation by β with a single pair of doubly asymptotic orbits, as discussed earlier. Let x be an arbitrary point of X and let y be any one of the p-points for Y. Let $\Pi(x, y) = y$. The extension $\Pi : X \times Y \to Y$ is distal, so as in the absolute case (x, y) is a relative p-point (and its image was chosen to be a p-point) but we assert (x, y) is not a p-point. To see this let x' be any point in X which is not on the orbit of x, and let y', y'' be distinct doubly asymptotic points in Y. Clearly $T^{n_i}(x', y') \to (x, y)$ iff $T^{n_i}(x', y'') \to (x, y)$.

We leave open the question as to how large a class of actions may be obtained beginning with the trivial flow and successively taking extensions which possess relative p-points or relative separators.

References

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